

Reply to Ternullo on the Multiverse

Doctor Ternullo raises a host of important issues, but I focus here on the central theme: his defense of the multiverse from the point of view of a naturalist, indeed, a naturalist of a particular variety that I call a 'second philosopher'. In a recent paper on the foundations of mathematics (a companion piece to my contribution to the present volume), I considered the possibility that some sort of multiverse theory could replace our current set theory in a range of foundational jobs now performed by ZFC + Large Cardinals (LCs). I concluded that for now, in the current state of knowledge, it isn't clear that this move is feasible or advisable. Ternullo apparently disputes this conclusion: 'the multiverse may be as acceptable as the universe for ... the foundational purposes of set theory' (Ternullo [201?], p. 3).

In an odd twist, though, Ternullo argues that one fundamental aspect of the foundational goal, the job I call Generous Arena,¹ is

¹ Ternullo also uses the term 'Generous Arena' for a different idea (Ternullo [201?], pp. 21-22), but in the passage under discussion here, he's concerned with the sense delineated in my contribution to this volume.

itself misguided, that the argument for it is, for my naturalist, 'slightly embarrassing' (Ternullo [201?], p. 20). As it happens, leading multiverse theorists don't see the situation this way; they embrace this foundational goal and argue that their theories meet it. For example, this from Hamkins:

The multiverse view does not abandon the goal of using set theory as an epistemological and ontological foundation for mathematics, for we expect to find all our familiar mathematical objects ... inside any one of the universes of the multiverse. (Hamkins [2012], p. 419)

And here's Steel:

We want one framework theory [i.e., foundational theory], to be used by all, so that we can use each other's work. It is better for all our flowers to bloom in the same garden. If truly distinct frameworks emerged, the first order of business would be to unify them. ... The goal of our framework theory is to *maximize interpretive power*, to provide a language and theory in which all mathematics, of today and of the future so far as we can anticipate it, can be developed. (Steel [2014], pp. 164-165)

Ternullo seems to reject this approach, on the grounds that the argument for the foundational goal itself is flawed (Ternullo [201?], pp. 19-20).

That argument, as he sees it, rests on the claim that Generous Arena was present early on in the history of set theory -- as it was -- and that this temporal priority implies logical priority. I think we can all agree that this is a weak argument, but it's not the right argument, as I hope is clear from §I of my contribution here.² Regardless of when they first arose, the mathematical attractions of Generous Arena, along with Risk Assessment, Metamathematical Corral,

² See §I of [2017] for a bit more detail.

and Shared Standard, remain as strong today as ever. In his willingness to forgo Generous Arena, Ternullo sounds a theme familiar, as we've seen, in category-theoretic and univalent foundations, but not one we hear from multiverse theorists like Hamkins and Steel. This creates some mismatch in the debate between Ternullo and me, since my analysis is explicitly addressed to the question of the multiverse's aptitude for filling this foundational role (among others) and he seems to think it needn't be filled. Still, there are some surrounding points well worth considering.

I begin, as Ternullo does, by attending to various versions of multiversism, though now with an eye not to the mathematical differences between them, but to the different philosophical or methodological stances one might take toward them. §II sketches my concerns about the multiverse in a foundational role, and §III considers other significant roles that multiverse thinking might play.

I. What is a multiverse view?

Ternullo begins with a straightforward characterization of the central distinction:

Whether set theory should be interpreted as the theory of a single universe of sets ... or ... as a theory about *multiple* structures ... that is, about a set theoretic 'multiverse'.
(Ternullo [201?], p. 1)

He alludes to Koellner's distinction between pluralism and non-pluralism:

pluralism ... maintains that ... although there are *practical* reasons that one might give in favor of one set of axioms over another -- say, that it is more useful for a given task --, there are no *theoretical* reasons that can be given ...

non-pluralism ... maintains that the independence results merely indicate the paucity of our standard resources for justifying mathematical statements. ... theoretical reasons can be given for new axioms. (Koellner [2014], p. 1)

Multiversism, then, is a model-theoretic version of pluralism:

There is not a single universe of set theory but rather a multiverse of legitimate candidates, some of which may be preferable to others for certain practical purposes, but none of which can be said to be the ‘true’ universe. (Koellner [2013], p. 3)

Ternullo agrees that multiversism is ‘an ontological (semantic) variant’ of pluralism (Ternullo [201?], p. 1). Presumably he also takes his multiversist to hold that, though there might be practical reasons for preferring one truth value for an indeterminate statement over another or one universe of the multiverse over another, there are no theoretical reasons for this -- there is no determinate truth value, no ‘true’ universe.

Now Ternullo notes that unabashedly metaphysical views like these may be problematic for the naturalist, but he concludes that the metaphysics can be disregarded (Ternullo [201?], §4.1). He has his own reasons for saying this, but in any case, we agree on the underlying point, that metaphysics is largely irrelevant to the mathematical issues at hand. In the hope of clarifying some of these matters, let me sketch a rough taxonomy of philosophical stances on multiversism, beginning from the most ontologically or semantically loaded and moving on from there.

On the deeply metaphysical end of the spectrum, there’s Hamkins’s position:

With forcing, we seem to have discovered the existence of other mathematical universes, outside our own universe, and the multiverse view asserts that yes, indeed, this is the case. (Hamkins [2012], p. 425) Each ... universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. (Ibid., pp. 416-417)

In Koellner's terms, presumably this Metaphysical Multiversism³ takes 'theoretical' considerations to tell us something about the structure of its generous ontology, perhaps, for example, that every world thinks ZFC.⁴ It might seem that the theoretical/practical distinction coincides with the intrinsic/extrinsic distinction familiar in the philosophical foundations of set theory -- where intrinsic considerations are somehow intuitive, or self-evident, or contained in the concept of set, and extrinsic considerations involve attractive consequences or interrelations or something of that sort -- but I think this can't be right. Koellner writes that ...

given the current state of our knowledge a case can be made for being a non-pluralist about ZFC and large cardinal axioms (Koeller [2013], p. 4)

... in other words, in multiverse terms, for assuming these axioms across all worlds of the multiverse, and from his other writings, it's

³ I use this term in place of Ternullo's 'ontological multiversism' to leave room for a position that replaces objective entities with determinate truth values.

⁴ Steel [2014] and Woodin [2011] both take ZFC to be true in every world of the multiverse, but Hamkins sometimes does not: 'There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF-, KP, and so on, perhaps even down to second-order number theory' (Hamkins [2012], p. 436). On the other hand, we've seen that he addresses the foundational goal like this: 'we expect to find all our familiar mathematical objects ... inside any one of the universes of the multiverse' (ibid., p. 419), which would seem to require at least ZFC (and if Risk Assessment is taken into account, large cardinals would be handy as well). In any case, Hamkins certainly embraces a number of objective truths ('multiverse axioms') about the multiverse in §9 of his [2012].

clear that some of this case is extrinsic. Similarly, Hamkins allows that

the mathematician's measure of a philosophical position may be the value of the mathematics to which it leads (Hamkins [2012], p. 440).

So it appears that at least some extrinsic considerations must also yield information about the multiverse, must also be included under 'theoretical'.

Suppose, then, that a certain set-theoretic statement, perhaps a candidate for a new axiom to be true in all worlds, or perhaps another sort of general claim about the structure of the multiverse, has many mathematical advantages and no mathematical disadvantages. This wouldn't be enough for the Metaphysical Multiversist to endorse it, because we'd have to be confident that those mathematical merits produce theoretical support, not mere practical support. We'd have to be confident that our belief in that set-theoretic statement, however attractive it might be, isn't just wishful thinking, that the objective mathematical realm of the multiverse doesn't just happen to deny us something we'd very much like to have. Many philosophers, in the tradition of Benacerraf's famous challenge to Platonism, would ask how we could come to know that our beliefs are tracking the truth about an abstract realm. My naturalist asks a question that's logically prior to Benacerraf's: why should we demand more than mathematical merits? Why should those mathematical merits be held hostage to extra-mathematical metaphysics? Her answer is that this is

wrong-headed, that the compelling mathematical reasons should be enough all by themselves.⁵

Though Metaphysical Multiversism is uncongenial to the naturalist, as Ternullo says, this isn't the end of the story; there are other varieties of multiversism. We could, for example, leave metaphysics aside and simply talk about theories. Set theory, then, isn't the project of describing an abstract mathematical realm; it's the project of forging a powerful mathematical theory to serve the foundation goal (among others).⁶ The universist advocates ZFC and its extensions in this role; the multiversist proposes an alternative multiverse theory of sets and worlds to take its place. For these purposes, all extrinsic considerations would be on equal footing; there'd be no distinction between 'theoretical' or truth-tracking cases and 'merely practical' cases. This Theory Multiversism is an option entirely open to the naturalist, should the evidence point that way.

A final variant sees the multiverse as analogous, not to a universe ontology, not to a universe theory, but to the iterative conception. In universe thinking, the iterative conception serves as an intuitive picture that helps us see our way around in deriving consequences from the axioms or seeking new avenues for axiom choice.

⁵ See [2011], pp. 55-59. In that book, I propose an alternative metaphysical position, Thin Realism, that avoids this problem by essentially reading its ontology off the analysis of proper methods, including extrinsic methods, but I doubt this is what Hamkins or Koellner or Ternullo has in mind.

⁶ We could think of this as the project of forming an optimally effective concept of set. Cf. the Arealism of [2011].

From the naturalist's non-metaphysical perspective, intrinsic considerations based on this picture are potentially of great heuristic value; the history of the subject amply demonstrates what an immensely successful tool the iterative conception has been. But, for the naturalist, it's important to stress that the value of this intuitive picture rests on the great mathematical merits of the work it's inspired, in other words, on its extrinsic success.⁷ If we were presented with an alternative intuitive picture that conceptualizes set-theory differently, if that alternative way of guiding the subject were more fruitful than the iterative conception, we should switch our allegiance without regret.⁸ The Heuristic Multiversist⁹ argues that the intuitive picture of a multiverse is just such an alternative; he might propose that ZFC and its extensions, guided by the iterative conception, should be replaced with a multiverse theory based on the new picture. This would be a version of Theory Multiversism, but other possibilities emerge in §III below. Either way, the basic suggestion is that the intuitive multiverse picture would guide the practice in new and different directions with important mathematical advantages.

There are no doubt other ways to frame a philosophical perspective for multiversism, and perhaps predictably, one prominent

⁷ See [2011], pp. 131-137.

⁸ Something like this actually happened when the intuitive picture of sets as extensions of properties fell out of favor in light of its conflict with the extremely fruitful axiom of choice.

⁹ Ternullo contrasts 'heuristic' with 'instrumental' ([201?], p. 11). See §III below.

multiverse theory, the one due to Steel, doesn't fit squarely in any of the three bins just described. As Ternullo notes, Steel is out to explore whether CH is 'meaningful'; his multiverse theory is intended, not as an alternative subject matter (Metaphysical Multiversism), not exactly as an alternative theory (Theory Multiversism), but as a way of determining which, if any, sentences in the language of set theory (not the multiverse language of sets and worlds) are 'meaningless', pose 'pseudo-questions'. How he goes about this and what conclusions he draws are quite subtle matters that go well beyond the scope of this reply.¹⁰ Still, I hope these three rough categories will help illuminate the debate between Ternullo and me. As this is an intramural debate between naturalists, we're focused primarily on multiversisms of the Theory and Heuristic varieties.

II. Naturalistic concerns about multiversism

In the paper Ternullo is discussing, I raise a number of questions about multiverse theories as potential alternatives to ZFC and its extensions as our basic foundational theory. The most fundamental of these is that a foundational theory, as we now understand it, has to be a theory, has to be an explicit set of axioms capable of doing the foundational jobs. Of the multiverse accounts on offer, only Steel's comes with a set of axioms, a fully explicit first-order theory of sets and worlds, but as noted, his goal is to

¹⁰ Toby Meadows and I hope to clarify some of these matters in 'A philosopher's guide to Steel's multiverse', in preparation. I also neglect the hyperuniverse program, simply because I don't understand it well enough to comment.

evaluate the sentences of ordinary set theory, not to replace them with something different. So, flat-footed as it sounds, the general lack of an explicit multiverse theory strikes me as a serious obstacle to a new and different multiverse foundation.

Hamkins's stand on the foundational status of the multiverse was quoted above:

We expect to find all our familiar mathematical objects ... inside any one of the universes of the multiverse. (Hamkins [2012], p. 419)

Roughly speaking, it seems any world of the multiverse can serve as our Generous Arena, and ZFC (satisfied by that world) as our Shared Standard. Presumably Risk Assessment is to be carried out in a world with large cardinals, that is, in ZFC+LCs. There's some question about Meta-mathematical Corral: if we only care about corralling a generous arena, we're once again thrown back on ZFC and its extensions; if we want to corral all of mathematics, it seems we'd need a theory of our multiverse, which we've seen Hamkins's doesn't provide. On Steel's view, ZFC+LCs turns up in the meaningful part of set-theoretic language and continues to carry out its usual foundational functions. For the most part, then, ZFC and its extensions retain their foundational roles -- in that respect, no alternative is actually on offer. So it's hard to see a case for replacing a universe view with multiverse view for foundational purposes.

But this isn't the end of the story. Some version of the multiverse perspective may have such attractive mathematical features that we're moved to adopt it even if a familiar theory like ZFC

remains our official foundation. Ternullo mounts a case along these lines.

III. Ternullo's defense

One striking turn in Ternullo's discussion is his characterization of Zermelo's famous 'On boundary numbers and domains of sets' (Zermelo [1930]) as 'the first description of a multiverse' (Ternullo [201?], p. 4).¹¹ If this were so, it would go a long way toward showing that multiversism has important and far-reaching mathematical consequences! Working in a strong implicit meta-theory, Zermelo presents an analysis of 'normal domains' characterized by second-order ZFC minus Infinity:¹² their 'boundary numbers' are inaccessible cardinals; they can be decomposed into ranks up to that number (this is touted as one of the extrinsic benefits of Foundation); any two with the same boundary number are isomorphic; for any two with different boundary numbers, one is an initial segment of the other. The question then arises: are there any normal domains, are there any boundary numbers? Zermelo mounts an argument in the meta-theory that for any ordinal α , there's a corresponding boundary number κ_α ; in modern terms, he's argued for the Axiom of Inaccessibles:

¹¹ To be clear, Ternullo isn't claiming that the historical Zermelo understood his work in multiverse terms (see Ternullo [201?], p. 5, footnote 8). He holds, rather, that Zermelo is a 'height potentialist' and that this position can be seen as a kind of 'height multiversism' (see footnote 13 below).

¹² He leaves out the axiom of infinity to allow for a 'finitary' normal domain acceptable to intuitionists (so for him ω is a boundary number). As he sees it, a generous store of normal domains makes set theory adaptable for a wide range of applications.

We must postulate the *existence of an unlimited sequence of boundary numbers* as a new axiom for the ‘meta-theory of sets’. (Zermelo [1930], p. 429)

Though Zermelo does take second-order ‘ZFC-Infinity’ to characterize each of an unending series of normal domains, I see no evidence that he intends his second-order ‘ZFC + a proper class of inaccessibles’ in the meta-theory as anything other than a description of the single universe in which all these normal domains reside.¹³ If including an axiom of inaccessibles is enough to qualify a list of axioms as a multiverse theory, then almost all set theorists these days are multiversists; this sets the bar far too low, renders the term useless. So it seems that Zermelo is best left out of this discussion.

Ternullo is on stronger ground when he extends the appeal to mathematical consequences into contemporary set theory. I’m happy to grant that, for example, Hamkins’s multiverse thinking has led to a fruitful investigation of ‘set-theoretic geology’ or that Steel’s approach has focused attention on important questions about the ‘core’. Cases like these display a clear heuristic benefit to thinking in terms of an intuitive multiverse picture -- on this Ternullo and I agree -- but he goes on to insist that these benefits aren’t purely heuristic, that they are actually ‘instrumental’. He draws this distinction from a question raised in my paper: can all

¹³ Many observers see Zermelo as a potentialist, but I have my doubts. Though everyone, actualist and potentialist alike, uses a familiar range of metaphors -- the universe is unending, etc. -- it seems to me that the cash value of potentialism is the rejection of quantification over all sets. But this is exactly what Zermelo seems to do, for example, in arguing that there’s an inaccessible for every ordinal.

the welcome mathematics inspired by multiverse thinking be carried out in our familiar universe theory, that is, are these all theorems of ZFC and its extensions? If the answer to this question is no, then multiverse thinking would be more than merely heuristic -- fully instrumental, in Ternullo's terms -- but as far as I can tell, the answer is yes, which Ternullo seems to acknowledge:

It has been made clear that there may be no specific task that the universist may not try to successfully emulate within their single V . However, it is, in my view, rather apparent that the multiverse enormously facilitates fundamental practical tasks. (Ternullo [201?], p. 24)

This is just to say that multiverse thinking is of great (purely) heuristic value.

Notice that we have here instances of Heuristic Multiversism different from what was envisioned in §I: the multiverse picture isn't being used to inspire new axioms toward a version of Theory Multiversism, but to inspire new mathematics, new concepts and methods, within our existing theory of ZFC and its extensions. And there's another potential contribution of Heuristic Multiversism, as well. Hamkins observes that

There is no reason to consider all universes in the multiverse equally, and we may simply be more interested in parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. (Hamkins [2012], p. 436)

Now the process of narrowing down to a restricted range of worlds may well be functionally equivalent to the process of adding new axioms to ZFC, so what's of interest here is the suggestion that thinking in multiverse terms could bring new and different considerations to bear on that process. In other words, multiverse thinking might help us to

refine our official theory of sets. In fact, this may be the ultimate upshot of Steel's line of thought: a stretch of multiverse thinking leads him to propose a new axiom for ordinary set theory.

In sum, then, our naturalist has no straightforward form of Theory Multiversism, only Steel's set of axioms with a different motivation, but there seems to be ample room for Heuristic Multiversism to do significant mathematical work in a number of different ways. We can draw two morals. The first is that ZFC and its extensions aren't uniquely tied to the intuitive universe picture of the iterative conception. They could be thought of, instead, as the shared theory of a range of worlds in the multiverse, so that what the universist sees as adding new axioms about V , could instead be seen as a narrowing of the range of worlds we take to be of interest. The second moral is one that should appeal to Ternullo's naturalism: since these intuitive pictures, universist and multiversist, are playing a merely heuristic role, there's no reason at all not to exploit them both, no reason at all not to switch back and forth depending on which is more suggestive in a given context. In the end, set theorists should feel entirely free to think in any intuitive terms that can lead them to good mathematics!

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